[T.K] 4.7.1.1 Let $a<b$. To show: $X \in U(a,b) \Leftrightarrow \varphi_X(t) := \mathbb{E}[e^{itX}] = \frac{e^{itb}-e^{ita}}{it(b-a)}$

$(\Rightarrow)$: $\varphi_X(t) = \mathbb{E}[e^{itX}] = \int_a^b e^{itx}\frac{1}{b-a}\,dx = \frac{1}{it(b-a)} e^{itx}\Big|_a^b = \frac{e^{itb}-e^{ita}}{it(b-a)}$.

$(\Leftarrow)$ By the course, $\varphi_X(t) = \varphi_Y(t)\ \forall t \in \mathbb{R} \Leftrightarrow X \overset{d}{=} Y$.

QED

[T.K] 4.7.1.2 Let $X \in Tri(-1,1)$, i.e $f_X(x) = \begin{cases} 1-|x| & \text{if } |x|<1 \\ 0 & \text{otherwise} \end{cases}$

To show: $\varphi_X(t) = \left(\frac{\sin(t/2)}{t/2}\right)^2$

$\varphi_X(t) = \mathbb{E}[e^{itX}] = \int_{-1}^{1} e^{itx}(1-|x|)\,dx = \int_{-1}^{0} e^{itx}(1+x)\,dx + \int_0^1 e^{itx}(1-x)\,dx$

$= \int_0^1 e^{-it\tilde{x}}(1-\tilde{x})\,d\tilde{x} + \int_0^1 e^{itx}(1-x)\,dx = \int_0^1 (e^{itx}+e^{-itx})(1-x)\,dx = \int_0^1 2\cos(tx)(1-x)\,dx$

$\tilde{x} = -x$ on the first integral.

by parts

$= 2\cdot\left\{ \underbrace{\frac{\sin(tx)}{t}(1-x)\Big|_0^1}_{=0} - \int_0^1 \frac{\sin(tx)}{t}\cdot(-1)\,dx \right\} = 2\int_0^1 \frac{\sin(tx)}{t}\,dx = \frac{2\cdot(-\cos(tx))}{t^2}\Big|_0^1 = \frac{1-\cos(t)}{t^2/2}$

Now notice that $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)\quad \forall x,y \in [0,2\pi]$

In particular, $\cos(x+y)+\cos(x-y) = 2\cos(x)\cos(y)\quad \forall x,y \in [0,2\pi]$

and if $x = y = t/2$, $\cos(t)+1 = 2\cos(t/2)^2 = 2(1-\sin(t/2)^2)$

$\Rightarrow \frac{1-\cos(t)}{t^2/2} = \frac{-2(1-\sin(t/2)^2)+2}{t^2/2} = \left(\frac{\sin(t/2)}{t/2}\right)^2$

QED

[T.K] 4.7.1.5 Let $n \geq 1$ and $X_1, X_2, \dots, X_n$ be iid $\mathcal{N}(0,1)$-distributed random variables.

To show: $\sum_{j=1}^n X_j^2 \in \chi^2(n)$.

Remmder: $X \in \chi^2(n)$, $n \in \mathbb{N}$ if $f_X(x) = \begin{cases} \frac{x^{\frac{n}{2}-1}e^{-x/2}}{\Gamma(\frac{n}{2})2^{n/2}} & \text{if } x>0 \\ 0 & \text{otherwise} \end{cases}$

First remark the following:

Let $t \in \mathbb{R}$, $\varphi_{\sum_{j=1}^n X_j^2}(t) = \mathbb{E}[e^{it\sum_{j=1}^n X_j^2}] = \mathbb{E}\left[\prod_{j=1}^n e^{itX_j^2}\right] \overset{X_j \text{ ind.}}{=} \prod_{j=1}^n \mathbb{E}[e^{itX_j^2}] \overset{X_j \text{ iid}}{=} \left(\mathbb{E}[e^{itX_1^2}]\right)^n$

Hence we will execute the following three steps :

(1.) Identify the distribution of X_1^2 and deduce its characteristic function.

(2.) Determine the expression of $\mathbb{E}[e^{it\sum_{j=1}^{n} X_j^2}]$

(3) Deduce $\sum_{j=1}^{n} X_j^2 \in \chi^2(n)$.

(1.) Two different techniques :

(a.) Moment-generating function : $\mathbb{E}[e^{tX_1^2}] = \int_{-\infty}^{+\infty} e^{tx^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}x^2(1-2t)} dx$ (*)

Now we want to find a constant C such that $C \cdot e^{-1/2 x^2(1-2t)}$ is the p.d.f of some normal distribution $\mathcal{N}(\mu,\sigma^2)$ to deduce the value of $\int_{-\infty}^{+\infty} e^{-\frac{1}{2}x^2(1-2t)} dx$.

$$C \cdot e^{-1/2 x^2(1-2t)} = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-1/2 \frac{(x-\mu)^2}{\sigma^2}} \Leftrightarrow \mu = 0 \text{ and } \sigma^2 = \frac{1}{1-2t} \Rightarrow \frac{1}{\sqrt{2\pi}\,\sigma} = \frac{\sqrt{1-2t}}{\sqrt{2\pi}} = C.$$

Hence $\frac{\sqrt{1-2t}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-1/2 x^2(1-2t)} dx = 1$ and $\int_{-\infty}^{+\infty} e^{-1/2 x^2(1-2t)} dx = \frac{\sqrt{2\pi}}{\sqrt{1-2t}}$

$\Rightarrow (*) = \frac{1}{\sqrt{1-2t}} \Rightarrow \mathbb{E}[e^{itX_1^2}] = \frac{1}{(1-2it)^{1/2}}$

(b.) Probability distribution function :

$$F_{X_1^2}(x) = \mathbb{P}(X_1^2 \le x) \underset{\sqrt{X_1^2} = |X_1|}{=} \mathbb{P}(|X_1| \le \sqrt{x}) = F_{X_1}(\sqrt{x}) - F_{X_1}(-\sqrt{x})$$

and $f_{X_1^2}(x) = \frac{d}{dx} F_{X_1^2}(x) = \frac{d}{dx}\{F_{X_1}(\sqrt{x}) - F_{X_1}(-\sqrt{x})\} = f_{X_1}(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} - f_{X_1}(-\sqrt{x}) \cdot \left(-\frac{1}{2\sqrt{x}}\right)$

$$= \frac{1}{2\sqrt{x}}\left(f_{X_1}(\sqrt{x}) + f_{X_1}(-\sqrt{x})\right) = \frac{1}{2\sqrt{x}}\left(\frac{1}{\sqrt{2\pi}} e^{-x/2} + \frac{1}{\sqrt{2\pi}} e^{-x/2}\right) = \frac{1}{\sqrt{2\pi}\sqrt{x}} e^{-x/2} \quad \forall x > 0.$$

$\Rightarrow X_1^2 \in \chi^2(1)$ and by the course, $\mathbb{E}[e^{itX_1^2}] = \frac{1}{(1-2it)^{1/2}}$

(2.) $\mathbb{E}[e^{it\sum_{j=1}^{n} X_j^2}] = \left(\mathbb{E}[e^{itX_1^2}]\right)^n = \frac{1}{(1-2it)^{n/2}}$

(3) By the course, $\frac{1}{(1-2it)^{n/2}}$ is the characteristic function of a $\chi^2(n)$-distributed random variable.

AED

[A.K] 4.7.1.8 (a) Let $X_1 \in N(0,1)$ and $X_2 \in N(0,1)$ be independent.

To show: $\varphi_{X_1 \cdot X_2}(t) = \frac{1}{\sqrt{1+t^2}} \quad \forall t \in \mathbb{R}$

$$\varphi_{X_1 \cdot X_2}(t) = \mathbb{E}[e^{it X_1 X_2}] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{itxy} f_{X_1}(x) f_{X_2}(y)\, dx\, dy = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{itxy} \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cdot \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dx\, dy \quad (*)$$

Reminder: if $X \in N(0,1)$, $\mathbb{E}[e^{itX}] = e^{-t^2/2}$ (this is proved in the course)

$$\Rightarrow (*) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \Big\{ \underbrace{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cdot e^{ityx}\, dx}_{\underset{t'=ty}{=} \mathbb{E}[e^{it'X_1}] = e^{-t'^2/2} = e^{-t^2y^2/2}} \Big\}\, dy = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} e^{-t^2y^2/2}\, dy = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2(1+t^2)}\, dy$$

$$= \int_{-\infty}^{+\infty} \frac{\sqrt{1+t^2}}{\sqrt{1+t^2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-y^2/2(1+t^2)}\, dy \underset{\sigma = \frac{1}{\sqrt{1+t^2}}}{=} \int_{-\infty}^{+\infty} \sigma \cdot \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{y^2}{2\sigma^2}}\, dy = \sigma = \frac{1}{\sqrt{1+t^2}} \quad \forall t \in \mathbb{R}$$

Recall: $X \in N(\mu, \sigma^2) \Leftrightarrow f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

QED

(b) Let $Z_1 \in \Gamma(a,b)$ and $Z_2 \in \Gamma(a,b)$ be two independent random variables. (b) Let Y has the distribution in (a). What are the values of a and b?

Set $U = Z_1 - Z_2$, suppose $U \overset{(d)}{=} Y$, where Y has the distribution in (a).

$$\mathbb{E}[e^{itU}] = \mathbb{E}[e^{it(Z_1 - Z_2)}] \overset{ind}{=} \mathbb{E}[e^{itZ_1}] \cdot \mathbb{E}[e^{-itZ_2}] \underset{\text{course}}{=} (1-bit)^{-a} \cdot (1+bit)^{-a} = (1+b^2t^2)^{-a}$$

and by assumption, $\mathbb{E}[e^{itU}] = \mathbb{E}[e^{itY}] = \frac{1}{\sqrt{1+t^2}} \Leftrightarrow b^2 = 1$ and $a = \frac{1}{2} \underset{a,b>0}{\Leftrightarrow} a = \frac{1}{2}$ and $b = 1$

QED